

BLOCK SYSTEMS OF FINITE DIMENSIONAL NON-COSEMISIMPLE HOPF ALGEBRAS

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ABSTRACT. In this paper, we study finite dimensional non-cosemisimple Hopf algebras through its underlying coalgebra. We decompose such a Hopf algebra as a direct sum of ‘blocks’. Blocks are closely related to each other by ‘rules’ and form the ‘block system’. Through the block system, we are able to give a lower bound for $\dim H$, where H is a non-cosemisimple Hopf algebras with no nontrivial skew-primitives, and a series of results about non-cosemisimple Hopf algebras of dimension $12p, 15p, 16p, 20p$ and $21p$ with p being a prime number.

INTRODUCTION

Throughout the paper, we fix an algebraically closed field \mathbb{k} with $\text{char } \mathbb{k} = 0$. Every vector space is finite dimensional over \mathbb{k} if not specified. For a coalgebra C , Δ , ϵ and $G(C)$ denote comultiplication, counit and the set of group like elements respectively. For a Hopf algebra, \mathcal{S} denotes the antipode.

The question of classifying all Hopf algebras of a given dimension comes from the Kaplansky’s ten conjectures. So far there are only a few general results. The classification splits into several different parts according to the coradical being cosemisimple, pointed or non-pointed non-cosemisimple. Recall that the coradical of a coalgebra is the direct sum of all its simple subcoalgebras. A coalgebra is called ‘cosemisimple’ if it equals its coradical, and it is called ‘pointed’ if every simple subcoalgebra is 1-dimensional. Hopf algebras which are both pointed and cosemisimple are group algebras. Then in the following discussions, by ‘pointed’ we mean ‘pointed but not cosemisimple’.

For a finite dimensional Hopf algebra, by [17], it is semisimple as an algebra if and only if it is cosemisimple as a coalgebra. There are a few important results on cosemisimple Hopf algebras, but there is no general strategy for their classification so far. For p, q, r being odd primes, it is already known that cosemisimple Hopf algebras with dimension $p, 2p, p^2, pq$ are trivial ([32],[19],[21],[11]), and the classification of cosemisimple Hopf algebras with dimension $p^3, 2p^2, pq^2, pqr$ are completed ([20],[18],[23],[24],[26],[12]).

At the moment, the most general method for the classification of pointed Hopf algebras is the lifting method developed by Andruskiewitsch and Schneider. First they decompose the associated graded Hopf algebra $gr(A)$ of a pointed Hopf algebra A into a smash biproduct

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of a braided Hopf algebra R and the group algebra $\mathbb{k}G(A)$, then study the structure of R as a Nichols algebra over $\mathbb{k}G(A)$, finally lift relations in R to get ones in A ([5],[3],[4],[6]).

For p, q being odd primes and $p < q \leq 4p + 11$, there is no non-cosemisimple non-pointed Hopf algebra with dimension $p, 2p, p^2, pq$ ([32],[29],[27],[30]). For non-cosemisimple non-pointed Hopf algebras of other dimensions, there are several classification methods based on different subjects. The dimension of spaces related to the coradical ([1],[7],[13]), the order of the antipode ([27],[28],[29],[30],[16]), the Hopf subalgebras and the quotient Hopf algebras ([25]) and the braided Hopf algebras in smash biproducts ([10]) all play important roles in the classification of non-cosemisimple non-pointed Hopf algebras.

The idea of block system is inspired by the classification technique used in [1], [7] and [13]. For a Hopf algebra, we build the block system based on its coradical filtration. Block systems become interesting when the Hopf algebras are non-cosemisimple and non-pointed. In this paper, we focus on non-cosemisimple Hopf algebras with no nontrivial skew primitives, which form a subclass of non-cosemisimple non-pointed Hopf algebras. Corollary 1.4, 1.5, 1.6, 1.8 and Proposition 1.10 are the ‘rules’ we know so far, which lead to our main results, Theorem 2.1 and Theorem 2.5. Theorem 2.1 shows a lower bound for the dimensions of non-cosemisimple Hopf algebras with no nontrivial skew primitives. By Radford’s formula for \mathcal{S}^4 , orders of group like elements are crucial in classification methods working with the antipode. So Theorem 2.5 contributes to the classification of Hopf algebras of dimensions involved in it.

The block system shows a way to understand a non-cosemisimple Hopf algebra through its coalgebra structure by decomposing it into blocks which are highly related by the ‘rules’. We believe that by further study, more ‘rules’ will be discovered, so more clearly we understand the Hopf algebra structure.

1. BLOCK SYSTEM

We first recall some useful results on dimensions of Hopf algebras related to the coradical filtration ([1, 7, 9, 13]).

For a coalgebra C , we denote by $\{C_n\}_{n \in \mathbb{N}}$ its coradical filtration.

For $x \in C$ and $g, h \in G(C)$, if $\Delta(x) = g \otimes x + x \otimes h$ and $x \notin \mathbb{k}(g - h)$, then x is called a *nontrivial skew primitives* of C .

By [22, Theorem 5.4.2], there exists a coalgebra projection $\pi : C \rightarrow C_0$ with kernel I . Set $\rho_L = (\pi \otimes id)\Delta$ and $\rho_R = (id \otimes \pi)\Delta$. Then C becomes a C_0 -bicomodule with the structure maps ρ_L and ρ_R .

Let P_n be the sequence of subspaces defined recursively by

$$\begin{aligned}
P_0 &= 0, \\
P_1 &= \{c \in C \mid \Delta(c) = \rho_L(c) + \rho_R(c)\} = \Delta^{-1}(C_0 \otimes I + I \otimes C_0), \\
P_n &= \{c \in C \mid \Delta(c) - \rho_L(c) - \rho_R(c) \in \sum_{1 \leq i \leq n-1} P_i \otimes P_{n-i}\}, \quad n \geq 2.
\end{aligned}$$

By [1, Lemma 1.1], $P_n = C_n \cap I$.

For a positive integer d , let D be a simple coalgebra of dimension d^2 . There is a *standard basis* $\{e_{i,j}\}_{i,j \in \{1, \dots, d\}}$ of D , such that

$$\Delta(e_{i,j}) = \sum_{k=1}^d e_{i,k} \otimes e_{k,j} \text{ and } \varepsilon(e_{i,j}) = \delta_{i,j}.$$

The j th column $\{e_{i,j}\}_{i \in \{1, \dots, d\}}$ is a basis of the simple left D -comodule $\bigoplus_{i \in \{1, \dots, d\}} \mathbb{k}e_{i,j}$ with comultiplication as comodule map. Every simple left D -comodule is isomorphic to $\bigoplus_{i \in \{1, \dots, d\}} \mathbb{k}e_{i,j}$. Similarly, the i th row $\{e_{i,j}\}_{j \in \{1, \dots, d\}}$ forms a basis of the unique simple right D -comodule up to isomorphisms.

Let \hat{C} be the set of isomorphism types of simple left C -comodules. There is a one to one correspondence between \hat{C} and the set of simple subcoalgebras of C . Every simple left C -comodule is isomorphic to the unique (up to isomorphisms) simple left comodule of a simple subcoalgebra of C . For $\tau \in \hat{C}$, let D_τ be the corresponding simple subcoalgebra of C with $\dim D_\tau = d_\tau^2$. Then we have

$$C_0 = \bigoplus_{\tau \in \hat{C}} D_\tau.$$

For convenience, we denote by D_g the simple subcoalgebra generated by $g \in G(C)$, and the index of D_g in \hat{C} is also g .

Set $C_{0,d} = \bigoplus_{\tau \in \hat{C}, d_\tau = d} D_\tau$ with $d \geq 1$. Easy to see that $\mathbb{k}G(C) = C_{0,1}$.

See that the dual of a simple left C -comodule is also a simple right C -comodule. For $\tau \in \hat{C}$, we denote by V_τ (respectively, V_τ^*) the simple left (respectively, right) C -comodule corresponding to the simple subcoalgebra D_τ (as a representative of the unique isomorphism type). Let $\{e_{i,j}^\tau\}_{i,j \in \{1, \dots, d_\tau\}}$ be a standard basis of D_τ , then

$$V_\tau \simeq \bigoplus_{i \in \{1, \dots, d_\tau\}} \mathbb{k}e_{i,j}^\tau \text{ and } V_\tau^* \simeq \bigoplus_{j \in \{1, \dots, d_\tau\}} \mathbb{k}e_{i,j}^\tau.$$

Since C_0 is cosemisimple and \mathbb{k} is a perfect field, any C_0 -bicomodule can be decomposed as a direct sum of simple C_0 -subbicomodules. It is well known that every simple C_0 -bicomodule is of the form $V_\tau \otimes V_\mu^*$ and of dimension $d_\tau d_\mu$, for some $\tau, \mu \in \hat{C}$ with $\dim V_\tau = d_\tau$ and $\dim V_\mu^* = d_\mu$.

Through ρ_L and ρ_R , C_n and P_n are C_0 -subbimodules of C . As in [7, 9, 13], for $\tau, \mu \in \hat{C}$, $P_n^{\tau, \mu}$ denotes the direct sum of simple C_0 -subbicomodules of the form $V_\tau \otimes V_\mu^*$ in P_n . $P_n^{\tau, \mu}$ is called *non-degenerate* if $P_n^{\tau, \mu} \not\subseteq P_{n-1}$. See that $P_n^{\tau, \mu}$ is a C_0 -bicomodule and $P_{n-1}^{\tau, \mu} \subseteq P_n^{\tau, \mu}$.

Then there exists a C_0 -bicomodule $Q_n^{\tau,\mu}$, which is also isomorphic to a direct sum of simple C_0 -bicomodules of the form $V_\tau \otimes V_\mu^*$, such that

$$P_n^{\tau,\mu} = P_{n-1}^{\tau,\mu} \oplus Q_n^{\tau,\mu}.$$

Then we give the definition of a block system.

Definition 1.1. For a coalgebra C , we set

$$B_n^{d_1,d_2} = \begin{cases} 0, & n = 0, d_1 \neq d_2, \\ C_{0,d}, & n = 0, d_1 = d_2 = d \geq 1, \\ \bigoplus_{\tau,\mu \in \hat{C}, d_\tau=d_1, d_\mu=d_2} Q_n^{\tau,\mu}, & n, d_1, d_2 \geq 1. \end{cases}$$

and call it a *block* of C .

It is obvious that $C = \bigoplus_{n \geq 0, d_1, d_2 \geq 1} B_n^{d_1,d_2}$, and we call this direct sum *the block system* of C .

Let H be a Hopf algebra. Follow the notations above. See that $P_n^{\tau,\mu}$ is non-degenerate if and only if $Q_n^{\tau,\mu} \neq 0$ for $\tau, \mu \in \hat{H}$ and $n \geq 1$. Hence we can rewrite some results in [9] and [13].

Proposition 1.2. For $\tau, \mu \in \hat{H}$, we have the following results.

- (1) [13, Lemma 3.2] If $Q_n^{\tau,\mu} \neq 0$ for some $n > 1$, then there exists a set of simple subcoalgebras $\{D_1, \dots, D_{n-1}\}$ such that $Q_i^{\tau,D_i} \neq 0$ and $Q_i^{D_i,\mu} \neq 0$ for all $1 \leq i \leq n-1$.
- (2) [13, Lemma 3.5] For $g \in G(H)$, $n \geq 1$, $\dim Q_n^{\tau,\mu} = \dim Q_n^{S\tau,S\mu} = \dim Q_n^{g\tau,g\mu} = \dim Q_n^{\tau g,\mu g}$, where the superscript $S\tau$ means that the simple subcoalgebra is $\mathcal{S}(D_\tau)$ and $g\tau$ (resp. τg) means the simple subcoalgebra gD_τ (resp. $D_\tau g$).
- (3) [13, Lemma 3.8] Assume that $Q_n^{\tau,\mu} \neq 0$ for some $n \geq 1$. If $d_\tau \neq d_\mu$ or $\dim Q_n^{\tau,\mu} \neq d_\tau^2$, then there exists a simple subcoalgebra E such that $Q_{n'}^{\tau,E} \neq 0$ for some $n' \geq n+1$.

Proposition 1.3. [9, Proposition 3.2 (i)] If H is a non-cosemisimple Hopf algebra with no nontrivial skew-primitives, then for any $g \in G(H)$, there exist $h \in G(H)$ and simple subcoalgebras D_τ and D_μ with $\tau, \mu \in \hat{H}$ and $d_\tau, d_\mu > 1$, such that $d_\tau = d_\mu$, $Q_1^{\tau,g} \neq 0$, $Q_k^{\tau,\mu} \neq 0$ for some $k > 1$ and $Q_m^{g,h} \neq 0$ for some $m > 1$.

The following results show properties of the blocks of a Hopf algebra, which should be considered as ‘rules’ for block systems.

Corollary 1.4. For blocks of H and $g \in G(H)$, we have

$$gB_n^{d_1,d_2} = B_n^{d_1,d_2}g = B_n^{d_1,d_2}, \text{ and } \mathcal{S}(B_n^{d_1,d_2}) = B_n^{d_2,d_1},$$

with $n \geq 0$ and $d_1, d_2 \geq 1$.

Proof. The image of a simple subcoalgebra of H under \mathcal{S} or left (right) action of $g \in G(H)$ is still a simple one of the same dimension. Hence the proposition is correct for $n = 0$.

\mathcal{S} , \mathcal{S}^{-1} and left action of $g \in G(H)$ are bijections and keep the coradical filtration. For $n \geq 1$, following the definition of $Q_n^{\tau,\mu}$, we have $gQ_n^{\tau,\mu} = Q_n^{g\tau,g\mu}$ and $\mathcal{S}(Q_n^{\tau,\mu}) = Q_n^{S\mu,S\tau}$. By Proposition 1.2(2), $gQ_n^{\tau,\mu} \subseteq B_n^{d_1,d_2}$ and $\mathcal{S}(Q_n^{\tau,\mu}) \subseteq B_n^{d_2,d_1}$ with $d_\tau = d_1, d_\mu = d_2$, which leads to $gB_n^{d_1,d_2} \subseteq B_n^{d_1,d_2}$ and $\mathcal{S}(B_n^{d_1,d_2}) \subseteq B_n^{d_2,d_1}$. Similarly $g^{-1}B_n^{d_1,d_2} \subseteq B_n^{d_1,d_2}$ and $\mathcal{S}^{-1}(B_n^{d_1,d_2}) \subseteq B_n^{d_2,d_1}$. Hence $gB_n^{d_1,d_2} = B_n^{d_1,d_2}$ and $\mathcal{S}(B_n^{d_1,d_2}) = B_n^{d_2,d_1}$.

Similar for $B_n^{d_1,d_2}g = B_n^{d_1,d_2}$. \square

Corollary 1.5. *For blocks of H , $\dim B_0^{1,1} = |G(H)|$ divides the dimension of every block of H .*

Proof. Let $B_n^{d_1,d_2}$ be a block of H . By Corollary 1.4, every block of H is stable under the left multiplication of $g \in G(H)$.

For $n = 0$, $B_0^{d_1,d_2}$ with $d_1 = d_2$ is a left $(H, \mathbb{k}G(H))$ -Hopf module with comultiplication of H as comodule map. Then by the Nichols-Zoeller Theorem, $B_0^{d_1,d_2}$ is a free left $\mathbb{k}G(H)$ -module. Then $|G(H)|$ divides the dimension of $B_0^{d_1,d_2}$.

For $n > 0$, we have

$$\begin{aligned} \Delta(B_n^{d_1,d_2}) &\subset H_0 \otimes (B_n^{d_1,d_2}) + (B_n^{d_1,d_2}) \otimes H_0 + \sum_{1 \leq i \leq n-1} P_i \otimes P_{n-i} \\ &\subset H_0 \otimes (B_n^{d_1,d_2}) + (B_n^{d_1,d_2}) \otimes H_0 + H_{n-1} \otimes H_{n-1}. \end{aligned}$$

See that H_{n-1} and $B_n^{d_1,d_2} \oplus H_{n-1}$ are left $(H, \mathbb{k}G(H))$ -Hopf modules with left multiplication of $\mathbb{k}G(H)$ as module map and comultiplication of H as comodule map. Then we have H_{n-1} and $B_n^{d_1,d_2} \oplus H_{n-1}$ are free over $\mathbb{k}G(H)$ and $|G(H)|$ divides the dimensions of H_{n-1} and $B_n^{d_1,d_2} \oplus H_{n-1}$, which leads to the fact that $B_n^{d_1,d_2}$ is free over $\mathbb{k}G(H)$ and $|G(H)|$ divides the dimension of $B_n^{d_1,d_2}$. \square

Every nonzero $Q_n^{\tau,\mu}$ leads to a nonzero block $B_n^{d_1,d_2}$ with $d_1 = d_\tau, d_2 = d_\mu$. Hence the following corollary is a direct result from Proposition 1.2.

Corollary 1.6. *For blocks of H , we have the following results.*

- (1) *If $B_n^{d_1,d_2} \neq 0$ for some $n > 1$, then there exists a set of positive integers $\{b_1, \dots, b_{n-1}\}$ such that $B_i^{d_1,b_i} \neq 0$ and $B_{n-i}^{b_i,d_2} \neq 0$ for all $1 \leq i \leq n-1$.*
- (2) *If $B_n^{d_1,d_2} \neq 0$ for some $n \geq 1$, then $B_n^{d_2,d_1} \neq 0$.*
- (3) *Assume that $B_n^{d_1,d_2} \neq 0$ for some $n \geq 1$. If $d_1 \neq d_2$, then there exists positive integers d_3 and d_4 such that $B_{l_1}^{d_1,d_3}, B_{l_2}^{d_4,d_2} \neq 0$ for some $l_1, l_2 \geq n+1$.*

Remark 1.7. Since we always assume that H is of finite dimension, then by Corollary 1.6(2),(3), $B_n^{d_1,d_2} \neq 0$ for $n \geq 1$ with $d_1 \neq d_2$ leads to $B_{n_1}^{d_1,d_1} \neq 0$ and $B_{n_2}^{d_2,d_2} \neq 0$ for some $n_1, n_2 \geq n+1$.

Corollary 1.8 is a direct result from Proposition 1.3 and Corollary 1.6 (2). It shows the necessary blocks for the block system of a non-cosemisimple Hopf algebra with no nontrivial skew-primitives.

Corollary 1.8. *If H is a non-cosemisimple Hopf algebra with no nontrivial skew-primitives, then there exist simple subcoalgebras D_τ and D_μ of H with $\tau, \mu \in \hat{H}$ and $d_\tau = d_\mu > 1$, such that $B_1^{d_\tau, 1}$, B_1^{1, d_τ} , $B_k^{d_\tau, d_\mu}$ and $B_m^{1, 1} \neq 0$ for some $k, m > 1$.*

Remark 1.9. By [1, Proposition 1.8], a Hopf algebra has no nontrivial skew-primitives if and only if it has no nontrivial pointed Hopf subalgebras. By nontrivial pointed Hopf subalgebras we mean pointed Hopf subalgebras which are not group algebras. Following [8, Lemma 2.8], if $(|G(H)|, \dim H/|G(H)|) = 1$, then H has no nontrivial skew-primitives. A special case is that $\dim H$ is free of squares.

For a pointed Hopf algebra, all the blocks are of the form $B_n^{1, 1}$ with $n \geq 0$. For a non-cosemisimple Hopf algebra with no nontrivial nontrivial skew-primitives, $B_n^{1, 1}$'s also play an important role.

Proposition 1.10. *For blocks of H , let $m = \max\{m' \mid B_{m'}^{1, 1} \neq 0\}$ and $l = \min\{m' \mid B_{m'}^{1, 1} \neq 0\}$, then*

- (1) $\dim B_m^{1, 1} = |G(H)|$;
- (2) *if $l < m$, and H is non-cosemisimple and has no nontrivial skew-primitives, then there exist $d_1, d_2, d_3, d_4 > 1$ and $l' > l > 1$, such that $B_{l'}^{d_1, 1} \neq 0$, $B_{l'}^{1, d_2} \neq 0$, $B_{l'-1}^{d_1, d_3} \neq 0$ and $B_{l'-1}^{d_4, d_2} \neq 0$.*

Proof. (1) If H is cosemisimple, then $m = 0$ and $B_0^{1, 1} = \mathbb{k}G(H)$.

If H is non-cosemisimple, then $m \geq 1$ by Corollary 1.8.

We built a basis for H at first.

For any $\sigma \in \hat{H}$, let $\{e_{i_\sigma, j_\sigma}^\sigma\}_{i_\sigma, j_\sigma \in \{1 \dots, d_\sigma\}}$ be a standard basis of D_σ .

Recall that

$$H = \bigoplus_{n' \geq 0, d_1, d_2 \geq 1} B_{n'}^{d_1, d_2} = \bigoplus_{d \geq 1} H_{0, d} \oplus \bigoplus_{n > 0, d_1, d_2 \geq 1} \left(\bigoplus_{\tau, \mu \in \hat{H}, d_\tau = d_1, d_\mu = d_2} Q_n^{\tau, \mu} \right),$$

where $Q_n^{\tau, \mu} = \bigoplus_k V_{n, k}^{\tau, \mu}$ with $V_{n, k}^{\tau, \mu} \simeq V_\tau \otimes V_\mu^*$ as simple H_0 -bicomodules for some finite index $k \in \{1, \dots, k(n, \tau, \mu)\}$. Here $k(n, \tau, \mu)$ denotes the number of simple H_0 -bicomodules of the form $V_\tau \otimes V_\mu^*$ in $Q_n^{\tau, \mu}$.

We have that D_τ is the simple subcoalgebra of H corresponding to V_τ with $\dim D_\tau = d_\tau^2$. Through the isomorphism from the unique simple left D_τ -comodule to V_τ , $\{e_{i_\tau, 1}^\tau\}_{i_\tau \in \{1 \dots, d_\tau\}}$ leads to a basis of V_τ . Similarly, $\{e_{1, j_\mu}^\mu\}_{j_\mu \in \{1 \dots, d_\mu\}}$ leads to a basis of V_μ^* . Then $\{e_{i_\tau, 1}^\tau \otimes e_{1, j_\mu}^\mu\}_{i_\tau \in \{1 \dots, d_\tau\}, j_\mu \in \{1 \dots, d_\mu\}}$ leads to a basis of $V_\tau \otimes V_\mu^*$, and through the isomorphism from $V_\tau \otimes V_\mu^*$ to $V_{n, k}^{\tau, \mu}$, we have a basis of $V_{n, k}^{\tau, \mu}$, denoted by $W_{n, k}^{\tau, \mu}$.

Set

$$W = \left(\bigcup_{\sigma \in \hat{H}, i_\sigma, j_\sigma \in \{1 \dots, d_\sigma\}} \{e_{i_\sigma, j_\sigma}^\sigma\} \right) \cup \left(\bigcup_{\tau, \mu \in \hat{H}, n > 0, k \in \{1, \dots, k(n, \tau, \mu)\}} W_{n, k}^{\tau, \mu} \right).$$

Then W is a basis of H .

Then we show that there is a basis element in $B_m^{1,1}$ such that its dual is the left integral of H^* .

Following notations at the beginning of this section, for any $w' \in W - H_0$, we have

$$\Delta(w') = \rho_L(w') + \rho_R(w') + \sum_{x,y \in W} k_{x,y} x \otimes y, \text{ for } k_{x,y} \in \mathbb{k}.$$

Set

$$L(w) = \begin{cases} \{x \mid k_{x,y} \neq 0, y \in W\}, & w \in W - H_0, \\ \phi, & w \in W \cap H_0, \text{ and} \\ R(w) = \begin{cases} \{y \mid k_{x,y} \neq 0, x \in W\}, & w \in W - H_0, \\ \phi, & w \in W \cap H_0. \end{cases} \end{cases}$$

There exists $u \in W \cap B_m^{1,1}$, such that $\rho_L(u) + \rho_R(u) = 1 \otimes u + u \otimes g$ for some $g \in G(H)$.

We claim that $u \notin L(w) \cup R(w)$ for all $w \in W$. If $u \in R(w)$, for some $w \in W$, then by the proof of [13, lemma 3.2], $w \in B_r^{d,1}$, for $r > m$. By Corollary 1.6 and Remark 1.7, $B_r^{d,1} \neq 0$ for $r > m$ leads to $B_{r'}^{1,1} \neq 0$ for $r' > m$, which is a contradiction with the maximality of m . Similar for $u \notin L(w)$.

Let $\{w^* \mid w \in W\}$ be the dual basis of W . Consider the convolution product $w^* * u^*$. For any $v \in W$, by $u \notin R(v)$ and the definition of W , we have,

$$w^* * u^*(v) = \begin{cases} 1 & w = 1, v = u, \\ 0 & \text{otherwise} \end{cases}.$$

Hence $w^* * u^* = w^*(1)u^*$ for $w \in W$. Then for any $f \in H^*$, $f * u^* = f(1)u^*$, which makes u^* a left integral of H^* .

Finally, we show that $\{hu\}_{h \in G(H)}$ is a basis of $B_m^{1,1}$.

See that $W \cap B_m^{1,1}$ is a basis of $B_m^{1,1}$. For $u' \in W \cap B_m^{1,1}$, assume that $\rho_L(u') + \rho_R(u') = g_1 \otimes u' + u' \otimes g_2$, $g_1, g_2 \in G(H)$. Then there exists $k \in \mathbb{k}$ such that $kg_1^{-1}u' \in W$ and

$$\rho_L(kg_1^{-1}u') + \rho_R(kg_1^{-1}u') = 1 \otimes kg_1^{-1}u' + kg_1^{-1}u' \otimes g_1^{-1}g_2.$$

Since the space of left integral of H^* is one-dimensional, we must have $kg_1^{-1}u' \in \mathbb{k}u$ and $g_1^{-1}g_2 = g$. Then $u' = k'g_1u$ for some $k' \in \mathbb{k}$. Since $\rho_L + \rho_R(\{hu\}_{h \in G(H)})$ is a linearly independent set in $H \otimes H$, $\{hu\}_{h \in G(H)}$ is also linearly independent. Then $\{hu\}_{h \in G(H)}$ is a basis of $B_m^{1,1}$ and $\dim B_m^{1,1} = |G(H)|$.

(2) Following notations in (1). Since H has no nontrivial skew-primitives, we have $B_1^{1,1} = 0$ and $l > 1$. By Corollary 1.6 (1), the existence of d_3, d_4 with $B_{l'-1}^{d_1, d_3} \neq 0$ and $B_{l'-1}^{d_4, d_2} \neq 0$ is ensured by the existence of d_1, d_2 .

For any $z \in W \cap B_l^{1,1}$, there exists $w \in W$ such that $z \in R(w)$ since $l < m$. Assume that $w \in B_t^{d,1}$.

If $d > 1$, set $d_1 = d_2 = d$ and $l' = t$. By Corollary 1.6 (2), $B_{l'}^{1, d_2} \neq 0$. Then such d_1, d_2 satisfy the requirements.

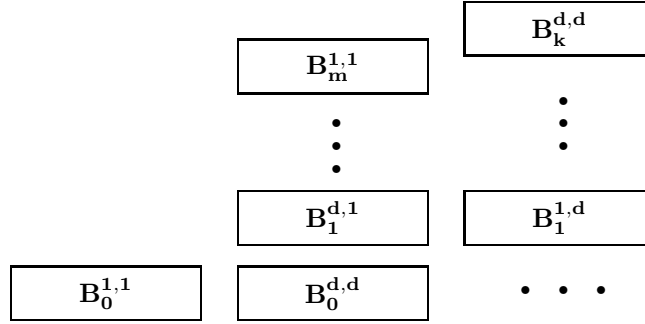
Otherwise $d = 1$. By the proof of [13, lemma 3.2] and the minimality of l , we have $t \geq 2l$. For $B_t^{1,1} \neq 0$, there exist $B_{t-1}^{1,d'}, B_1^{d',1} \neq 0$. Since H has no nontrivial skew-primitives, we have $d' > 1$. With $t-1 \geq 2l-1 > l$, $B_{t-1}^{1,d'}$ and $B_{t-1}^{d',1}$ are not zero. Set $d_1 = d_2 = d'$ and $l' = t-1$. Then (2) is proved. \square

2. APPLICATIONS

By Corollary 1.6 and Corollary 1.8, if H is a non-cosemisimple Hopf algebra with no nontrivial skew-primitives, then its block system has the form as

$$B_0^{1,1} \oplus B_0^{d,d} \oplus B_1^{d,1} \oplus B_1^{1,d} \oplus \cdots \oplus B_m^{1,1} \oplus B_k^{d,d},$$

for some $d > 1$, $k = \max\{k' \mid B_{k'}^{d,d} \neq 0\}$ and $m = \max\{m' \mid B_{m'}^{1,1} \neq 0\}$. By Remark 1.7, such a k exists and if $B_n^{d,d'} \neq 0$ for some $d' \geq 1$, then $n \leq k$. We use a diagram (Figure 1) to show the structure of the block system.



(Figure 1)

These six blocks are necessary for every non-cosemisimple Hopf algebra with no nontrivial skew-primitives. If the block system of a coalgebra does not contain these six blocks above, then this coalgebra does not admit the structure of a non-cosemisimple Hopf algebra with no nontrivial skew-primitives.

We find a lower bound for the dimension of a non-cosemisimple Hopf algebra with no nontrivial skew-primitives, which is a generation of [9, Proposition 3.2]. For $n_1, n_2 \in \mathbb{N}$, we denote by $\text{lcm}(n_1, n_2)$ the least common multiple of n_1 and n_2 .

Theorem 2.1. *If H is a non-cosemisimple Hopf algebra with no nontrivial skew-primitives and $|G(H)| = r$, then*

$$\dim H \geq \min\{(2d+2)r + 2\text{lcm}(d^2, r) \mid d > 1\}.$$

Proof. First we take the lower bound of the possible dimensions of each necessary block. Then add them together (see that these sums are related to d). Finally pick the minimal one for all $d > 1$ as the lower bound of $\dim H$.

We have $\dim B_0^{1,1} = |G(H)| = r$. Since n divides the dimension of every block and d^2 divides $\dim H_{0,d}$, we have $\dim B_0^{d,d} = \dim H_{0,d}$ is a multiple of $\text{lcm}(d^2, r)$. Similarly $\dim B_k^{d,d}$ is a multiple of $\text{lcm}(d^2, r)$. By Proposition 1.10 (1), $\dim B_m^{1,1} = |G(H)| = r$.

The dimension of a simple H_0 -bicomodule in $B_1^{d,1}$ is d , and with left multiplications of group like elements in $G(H)$, we have another $|G(H)| - 1$ simple H_0 -bicomodules of different isomorphism types in $B_1^{d,1}$. Hence $\dim B_1^{d,1}$ is a multiple of $d|G(H)| = dr$. By Corollary 1.4, $\dim B_1^{1,d} = \dim B_1^{d,1}$ is also a multiple of dr . \square

For further discussions, we make some notations at first.

Definition 2.2. For $r, d_1, d_2 \geq 1$,

(1) set

$$f(r, d_1, d_2) = \begin{cases} r & d_1 = 1, d_2 = 1, \\ 2d_2r & d_1 = 1, d_2 \neq 1, \\ 2d_1r & d_1 \neq 1, d_2 = 1, \\ \text{lcm}(d_1d_2, n) & d_1 \neq 1, d_2 \neq 1; \end{cases}$$

(2) set $L_{r,d_1,d_2} = \mathbb{k}^{\oplus f(r,d_1,d_2)}$ as a \mathbb{k} -space of dimension $f(r, d_1, d_2)$ and call it a *basic block*;

(3) for $d > 1$, set $L(r, d) = (L_{r,1,1})^{\oplus 2} \oplus (L_{r,d,d})^{\oplus 2} \oplus L_{r,d,1}$ and call it a *minimal form*.

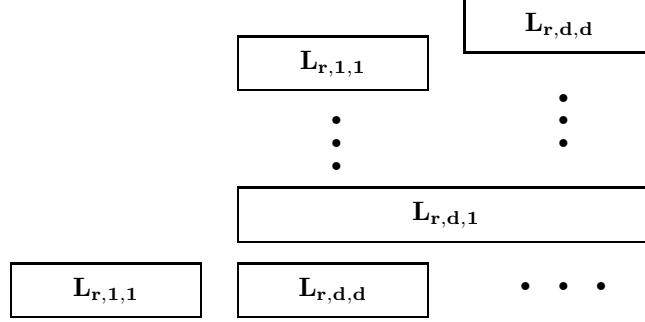
For $|G(H)| = r$ and $d, d_1, d_2 > 1$, $n \geq 0$, we have

- (1) $\dim L(r, d) = (2d + 2)r + 2\text{lcm}(d^2, r)$;
- (2) $\dim B_n^{1,1}$ is a multiple of $r = \dim L_{r,1,1}$;
- (3) $\dim B_n^{d,1} \oplus B_n^{1,d}$ is a multiple of $2dr = \dim L_{r,d,1} = \dim L_{r,1,d}$;
- (4) $\dim B_n^{d_1,d_2}$ is a multiple of $\text{lcm}(d_1d_2, r) = \dim L_{r,d_1,d_2}$.

Then as \mathbb{k} -spaces, $B_n^{1,1}$, $B_n^{d,1} \oplus B_n^{1,d}$ and $B_n^{d_1,d_2}$ are isomorphic to a multiple of $L_{r,1,1}$, $L_{r,d,1}$ and L_{r,d_1,d_2} , respectively.

Assume that H is non-cosemisimple and has no nontrivial skew-primitives. Then as \mathbb{k} -spaces, the block system of H is isomorphic to a minimal form $L(|G(H)|, d)$ with more (or no) basic blocks being added. Of course, the adding of basic blocks into the minimal form must coincide with ‘rules’ of block system.

Compare Figure 1 and Figure 2 to see the corresponding relations between necessary blocks and basic blocks.



(Figure 2)

With more condition on $|G(H)|$, we have the next result following Proposition 1.10 and Theorem 2.1. See that $|G(H)|$ divides $\dim H$.

Corollary 2.3. *If H is a non-cosemisimple Hopf algebra with no nontrivial skew-primitives and $|G(H)| = p$ with p being a prime number, set $\dim H = tp$ with $t \geq 1$, then we have*

- (1) *if $p = 2$, then $t \neq 1, 2, \dots, 9, 11, 13, 15$;*
- (2) *if $p = 3$, then $t \neq 1, 2, \dots, 13, 15, 16, 19$;*
- (3) *If $p > 3$, then $t \neq 1, 2, \dots, 13, 15, 16, 17, 19$ (if $p \neq 5$), $20, 21$ (if $p \neq 7$).*

Proof. We only prove (2) here, since the proof for (1), (2) and (3) are similar.

In (2), with $p = 3$, the lower bound of $\dim H$ by Theorem 2.1 is $14p$, which corresponds to two different minimal forms $L(p, 2)$ and $L(p, 3)$. Then $t \neq 1, 2, \dots, 13$ are proved.

See that $L(p, 4) = 42p$. Hence, if $14p < \dim H < 42p$, then as \mathbb{k} -spaces, the block system of H must be isomorphic to $L(p, 2)$ or $L(p, 3)$ with basic blocks added.

If $t = 15, 16$, or 19 , it is not difficult to see that there are at least one basic block $L_{p,1,1}$ of $\dim p$ being added.

By Proposition 1.10 (1), the block $B_m^{1,1}$ with $m = \max\{m' \mid B_{m'}^{1,1} \neq 0\}$ has a fixed dimension p . This means the new added basic block $L_{p,1,1}$ leads to a nonzero block $B_n^{1,1}$ of H with $n < m$. Then for $l = \min\{m' \mid B_{m'}^{1,1} \neq 0\}$, $l < m$. By Proposition 1.10 (2), there exists $B_{l'}^{1,d} \neq 0$, $B_{l'}^{d,1} \neq 0$ and $B_{l'-1}^{d,d} \neq 0$ for some $l' > l > 1$ and $d > 1$. Each of the three blocks is not one of the six necessary blocks of H . Then there are more basic blocks need to be added into $L(p, 2)$ or $L(p, 3)$.

For $L(p, 2)$, $\dim H \geq \dim L(p, 2) + \dim L(p, 1, 1) + \dim L(p, 2, 1) + \dim L(p, 2, 2) = 21p$; for $L(p, 3)$, $\dim H \geq \dim L(p, 3) + \dim L(p, 1, 1) + \dim L(p, 3, 1) + \dim L(p, 3, 3) = 24p$. Both contradict with $\dim H = tp$. So (2) is proved. \square

Corollary 2.3 provides a new angle to look at things we have already known. For example, the fact proved in [14] that there is no non-cosemisimple Hopf algebra of dimension

30 is a direct result from this corollary, Theorem 2.1 and [29, Corollary 2.2]. If H is a noncosemisimple Hopf algebra with $\dim H = 30$, by Theorem 2.1, $|G(H)| \neq 15, 10, 6$, and by Corollary 2.3, $|G(H)| \neq 5, 3, 2$. By [29, Corollary 2.2], H or H^* is not unimodular. Then $|G(H^*)| = 30$ or $|G(H)| = 30$, which contradicts with H being non-cosemisimple.

Remark 2.4. Some of the results in Corollary 2.3 have already been proved by different methods.

Under conditions of Corollary 2.3, let q be a prime number with $p < q$. By [29], there is no non-cosemisimple Hopf algebra of dimension $2q$. By [2], a non-cosemisimple Hopf algebra of dimension p^2 is a Taft algebra, which is pointed. By [28] and [30], Hopf algebras of dimension pq with p, q being odd primes and $p < q \leq 4p + 11$ are cosemisimple. Then all cases in (1)(2)(3) that t does not equal a prime number are covered.

For $p = 2$, $t \neq 4$ is proved in [31], which shows that there is no non-cosemisimple non-pointed Hopf algebra of dimension 8; $t \neq 6$ is proved in [25] that non-cosemisimple Hopf algebras of dimension 12 always have nontrivial primitives; $t \neq 8$ is proved in [15], in which Hopf algebras of dimension 16 are classified completely; $t \neq 9$ is proved in [16], which shows that there is no non-cosemisimple non-pointed Hopf algebra of dimension $2q^2$; and $t \neq 15$ follows from [14], which completes the classification of Hopf algebras of dimension 30.

For $p = 3$, $t \neq 4, 6$ follows from [25] and [16], and $t \neq 8, 9, 10$ is well known by [9, Proposition 3.2].

Following Remark 1.9, we list results in Corollary 2.3 which have not been given by others as far as we know:

Theorem 2.5. *If H is a non-cosemisimple Hopf algebra, then we have the following results.*

- (1) *For a prime number p ,*
 - (a) *if $\dim H = 12p$ with $p > 3$, then $|G(H)| \neq p$;*
 - (b) *if $\dim H = 15p$ with $p > 5$, then $|G(H)| \neq p$;*
 - (c) *if $\dim H = 16p$ with $p \geq 3$, then $|G(H)| \neq p$;*
 - (d) *if $\dim H = 20p$ with $p > 5$, then $|G(H)| \neq p$;*
 - (e) *if $\dim H = 21p$ with $p = 5$ or $p > 7$, then $|G(H)| \neq p$.*
- (2) *Assuming that H has no nontrivial skew-primitives,*
 - (a) *if $\dim H = 36$ or 45 , then $|G(H)| \neq 3$;*
 - (b) *if $\dim H = 75$ or 100 , then $|G(H)| \neq 5$.*

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